$$l'=g'/\pi^2=\frac{32.2015235}{9.8696044}(1+\tfrac{1}{4}e^2\sin^2\theta)=3.2626965(1+\tfrac{1}{4}e^2\sin^2\theta) \text{ feet,}$$

$$=39.1523578(1+\tfrac{1}{4}e^2\sin^2\theta) \text{ inches}=994.45(1+\tfrac{1}{4}e^2\sin^2\theta)mm.$$

$$\therefore l'=L'(1+\tfrac{1}{4}e^2\sin^2\theta).$$

$$l=g/\pi^2=\frac{32.0902682}{9.8696044}(1+\tfrac{3}{4}e^2\sin^2\theta)=3.251424(1+\tfrac{3}{4}e^2\sin^2\theta) \text{ feet}$$

$$=39.017088(1+\tfrac{3}{4}e^2\sin^2\theta) \text{ inches}=991.01(1+\tfrac{3}{4}e^2\sin^2\theta)mm.$$

$$\therefore l=L(1+\tfrac{3}{4}e^2\sin^2\theta).$$

To find the time of revolution of the earth on its axis so that centrifugal force may equal gravity.

First, at the equator.

Let A be the bob of a conical pendulum, $DC=b=$height of cone generated by this pendulum. The formula for time of revolution of this pendulum is

$$t=2\pi\sqrt{\frac{b}{G'}}=2\pi\sqrt{\frac{12b}{\pi^2L'}}=2\sqrt{\frac{12b}{L'}}$$

where L' is given in inches, and b in feet.

$$\therefore t=2\sqrt{\frac{20851005}{3.2626965}}=5056 \text{ seconds.}$$

At present the earth revolves once 86164 seconds. $86164\div5056=17.04$ times as fast as at present.

Second, in latitude 30°.

Let P be the bob of the pendulum.

Then $t=2\pi\sqrt{\frac{DH}{g'}}=2\sqrt{\frac{12DH}{l'}}$, l' in inches, DH in feet.

$$DH=b-CH=b-r\sin\varphi=b-\frac{b\sqrt{(1-e^2)}\sin\theta}{\sqrt{(1-e^2\sin^2\theta)}}=b\left(1-\sqrt{\frac{1-e^2}{4-e^2}}\right), \text{ since } \theta=30°.$$

$$\therefore DH=.5013b=10452608.8065 \text{ feet.}$$

$$l'=39.1523578(1+\tfrac{1}{16}e^2)=39.16929336 \text{ inches.}$$

$$\therefore t=2\sqrt{\frac{10452608.8065}{3.26410778}}=3579 \text{ seconds.}$$

$86164 \div 3579 = 24.075$ times as fast as at present.

$\therefore$ One revolution per hour.

By this method we can find the time of revolution for any latitude. It is

$$t = 4\sqrt{\frac{3b}{L'}}\left\{\frac{\left(1-\frac{\sqrt{(1-e^2)}\sin\theta}{\sqrt{(1-e^2\sin^2\theta)}}\right)}{(1+\frac{1}{4}e^2\sin^2\theta)}\right\}^{\frac{1}{2}}$$

where b is taken in feet, and L' in inches.

The following table gives gravity and the length of the second's pendulum for every five degrees under existing conditions.

DEGREES	GRAVITY		SECOND'S PENDULUM	
	POUNDALS	DYNES	INCHES	*mm*
0°	32.0002682	978.09	39.017088	991.01
5°	32.0915197	978.13	39.018610	691.05
10°	32.0952903	978.25	39.023194	991.17
15°	32.1014356	978.43	39.030666	991.36
20°	32.1097470	978.69	39.040771	991.62
25°	32.1200159	979.00	39.053257	991.93
30°	32.1319053	979.36	39.067713	992.30
35°	32.1450623	979.76	39.083710	992.71
40°	32.1590697	980.19	39.100741	993.14
45°	32.1735424	980.63	39.118337	993.59
50°	32.1880023	981.07	39.135918	994.03
55°	32.2020226	981.50	39.152965	994.47
60°	32.2151796	981.90	39.168962	994.87
65°	32.2270722	982.26	39.183422	995.24
70°	32.2373379	982.57	39.195903	995.56
75°	32.2456621	982.83	39.206024	995.81
80°	32.2517955	983.02	39.213482	996.00
85°	32.2555651	983.13	39.218065	996.12
90°	32.2568167	983.17	39.219587	996.16

A PROBLEM CONNECTED WITH MERSENNE'S NUMBERS.

By HARRY S. VANDIVER, Bala, Pa.

In Sir W. R. Ball's *Recreations and Problems* (London, 1809), page 33, I find the following:

"A curious proposition which comes from China, and which I believe appears here in print for the first time, is *that* $(2^n-2)/n$ *is an integer if* n *is a prime and is not an integer if* n *is not a prime.* The first of these statements is at once demonstrable; but I have not succeeded in proving the second part of the propo-

sition, which seems to introduce considerations similar to those involved in the theory of Mersenne's Numbers."

This proposition is of extraordinary importance, since, if we suppose it true then it gives a complete analytic definition of a prime. For instance, to find whether or not a is a prime it would only be necessary to calculate the residue of 2^{a-1} with respect to a. If we find $2^{a-1}\equiv 1 (\mathrm{mod} a)$ then a is a prime, but if 2^{a-1} is not congruent to 1 $(\mathrm{mod} a)$ then a is composite.

The object of the following investigation is to show that the second part of the proposition is *false*.

To prove this falsity it is sufficient to find a value n=an odd composite such that $2^{n-1}\equiv 1(\mathrm{mod} n)$.....(1).

Let us suppose, first that n is the simplest form of odd composite$=p\times q$, where p and q are primes. Then, since $\phi(n)=(p-1)(q-1)$, we have

$$2^{(p-1)(q-1)}\equiv 1(\mathrm{mod} pq)$$

by Fermat's Generalized Theorem. If we assume that m is the smallest number such that $2^m\equiv 1(\mathrm{mod} pq)$, then m must be a divisor of $\phi(n)$ and if (1) is possible then n must be a multiple of m. (Serret's *Algebra*, Sup. Vol. 2, page 48).

These conditions may be written

$$pq-1\equiv 0(\mathrm{mod} m)$$
$$(p-1)(q-1)\equiv 0(\mathrm{mod} m)$$

Comparing these congruences, we find

$$p\equiv 1(\mathrm{mod} m),\quad q\equiv 1(\mathrm{mod} m).$$

Then to prove the possibility of (1) it is sufficient to find values, m, p, q, such that $2^m\equiv 1(\mathrm{mod} pq)$ where $q\equiv 1(\mathrm{mod} m)\equiv q$ and m is not greater than $\phi(pq)$.

To find these values, assign to m small integral values in succession to find, by trial, numbers for p and q corresponding.

If m=1 to 9 inclusive, no appropriate values can be found for p and q, but if m=10, then we can put

$$p=10\times 1+1 \text{ and } q=3\times 10+1,$$

and then $$2^{10}\equiv 1(\mathrm{mod} 31\times 11)$$

and therefore $$2^{340}\equiv 1(\mathrm{mod} 31\times 11)$$

and the second part of the proposition originally quoted is thus seen to be false.

If m=11, then putting p=23, q=89,

$$2^{2046}\equiv 1(\mathrm{mod} 2047).$$

In the same manner we can find other values of m, p, and q, and the number of possible sets does not appear to be limited.

The first part of the proposition, namely, $(2^n-2)/n$ is an integer when n is prime, is but a particular case of Fermat's Theorem that $a^{p-1}\equiv 1 \pmod{p}$ when p is prime and a is prime to p.

Bala, Pa., Feb. 1, 1902.

GEOMETRIC DERIVATION OF CERTAIN TRIGONOMETRIC FORMULAE.

By PROFESSOR L. E. DICKSON.

Students of trigonometry find it interesting to have, in addition to the usual proof, the following geometric derivation of the formulae used in the solution of a plane triangle of given sides. The only trigonometry used is the *definition* of the tangent ratio.

The first step is the usual geometric proof (by means of the theorem giving, in terms of the sides, the projection of one side on another side) of Heron's formula:

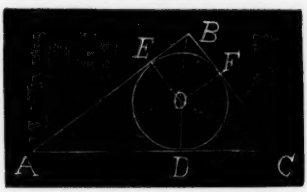
$$\triangle = \sqrt{[s(s-a)(s-b)(s-c)]}.$$

The next step is the evaluation of the radius r of the circle inscribed in the triangle. It its center is O, the areas of the triangles AOB, BOC, COA are $\frac{1}{2}cr$, $\frac{1}{2}ar$, $\frac{1}{2}br$, respectively. Hence

$$\triangle = \tfrac{1}{2}(a+b+c)r = sr.$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

The next step is the simple proof that the length of the tangent from A to the inscribed circle is $AE=AD=s-a$; from B, $BE=BF=s-b$, and from C, $CF=CD=s-c$. Then



$$\tan\tfrac{1}{2}A = \frac{r}{s-a}, \quad \tan\tfrac{1}{2}B = \frac{r}{s-b}, \quad \tan\tfrac{1}{2}C = \frac{r}{s-c}.$$

These are the most convenient formulae for the solution of a triangle of given sides. We may, however, derive at once the formulae for $\tan\frac{1}{2}A$, $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$ in terms of a, b, c only. Evidently,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)\ (s-c)}{s(s-a)}}.$$

We may obtain $\cos\frac{1}{2}A$ by employing the formula

$$\sec^2\tfrac{1}{2}A=1+\tan^2\tfrac{1}{2}A=1+\frac{(s-b)\ (s-c)}{s(s-a)}=\frac{2s^2-s(a+b+c)+bc}{s(s-a)}=\frac{bc}{s(s-a)};$$

or from the figure, since $AO^2=r^2+(s-a)^2$ reduces to $bc(s-a)/s$. Then

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}},\ \sin\tfrac{1}{2}A=\tan\tfrac{1}{2}A\cos\tfrac{1}{2}A=\sqrt{\frac{(s-b)\ (s-c)}{bc}}.$$

Similarly, a proof of the sine proportion follows from the relation $a/\sin A=D$=constant=diameter of circumscribed circle.

University of Chicago, Jan. 1902.

SOME FALLACIES IN WENTWORTH'S GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

Often both to teacher and pupil instruction seems satisfactory which to the expert is deliciously absurd.

The American-University-girl's French in Paris is a field of rich humor. Fashionable restaurants have hired waiters for their ability to understand this strange language, American French.

I went once with a party of Americans to buy gloves at a famous place on the Avenue de la Opera. "How does it happen that you understand our French so well?" was asked the attendant.

"Why, that's not strange," she answered, "I am an English-woman!"

Again, a young lady from Texas, a student of French, bought a parasol at the Bon Marché, that store which has the genius to say "We will exchange what you buy, whenever it ceases to please you." The very next day she took it back, but what she really did say in French to the attendant was "I bought this here *tomorrow*." The polite attendant never smiled, but, though he spoke English fluently, answered slowly in French: "If mademoiselle bought it here tomorrow, we are glad to receive it back or exchange it."

The French words for horses (chevaux) and for hair (cheveux) differ only in the one vowel, and Americans are vowel deaf.

This explains the apparently mysterious statement of the young *coiffeur* who said, "I begin to learn that American French. When an American lady drives up and asks me to wash her horses, that means she wishes her hair dressed."

A gushing southern young lady who felt sure of the word 'fiancé' for 'engaged', astounded a young hack-man by asking him in French, "Are you betrothed?" and on his answering "no," said: "Then I take you." He was so surprised that when asked his fare, always a mistake in Paris, he told the truth, but when it came to pay, asked thrice as much; when the girl exclaimed in American French, "You are more dear to me now than when we were first engaged."

Our books in elementary mathematics are American French, deliciously absurd to the expert.

Almost the only geometry used is that of my friend the New Hampshire banker, Wentworth, who begins with a mental confusion between a terminated piece of a straight line, which completely satisfies his newest definition for a straight line, and the straight line itself. This confusion is emphasized by his "Postulate: A straight line can be produced."

If this is what he means, it is a definite magnitude, like an inch or a yard. But then about it his axiom is not true.

Axiom: "Two points *determine* a straight line."

On the same page 8, is "§49. Axiom. A straight line is the shortest line that can be drawn from one point to another."

But as the great Hilbert well says: "Such an axiom is senseless, if one has not defined the concept of length of a curve."

So it need not surprise us that in attempting to attain to the length of the simplest curve, the circle, Mr. Wentworth falls into a shocking *non-sequitur*, which blunder is now on page 95 of his revised edition, 1899, and is being duly taught to thousands upon thousands of our young Americans, and by them duly swallowed as mathematics.

On page 9 he defines an angle as an *opening* [the italics are his].

On page 12, he is evidently standing on his head, since he talks upside down when he says "the magnitude of the angle depends upon the amount of rotation of the line."

But anyone who even contemplates owning a watch should have realized that it is the angle which determines the amount of rotation.

The demonstrations of even his simplest theorems are fallacious or beg the question.

Thus to prove "In an isosceles triangle the angles opposite the equal sides are equal," he uses a hypothetical construction on page 36 which he does not justify by any reference for the obvious reason that such reference would be to page 115 of his own book.

In what he gave as proof of the theorem: "Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other," up to 1886 he said, page 44, "place triangle $A'B'C'$ in the position $AB'C$, having its *greatest* side $A'C'$ in coincidence with its equal AC, and its vertex at B', opposite B. Draw BB' intersecting AC at H."

Of course he did not prove this intersection, since it is enormously harder to prove than the whole theorem. But its standing unjustified must have been a

source of trouble, since in his later editions it is simply omitted, though as before his proof depends upon it.

His treatment of tangent circles has always been consistently fallacious. Two circles are called tangent when they have only one point in common. It follows then as a theorem that the line of centers passes through the point of contact. Wentworth has this theorem, page 91, but proves it by assuming it in his definition, page 75, §221. "Two circles are tangent to each other, *if both are tangent to a straight line* at the same point," which is of course only another form of the corollary: a perpendicular to the center-straight through the point of contact is a common tangent to the two circles.

The rest of the book is equally vulnerable, but, not thrice to slay the slain, we desist.

University of Texas.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

An operator on 'Change gains 5% on his *daily capital* every *odd* day of a business-week, and loses 5% of the same capital every *even* day of same week. What per cent. of his *original capital* will he have gained, or lost, at the end of a business-week?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; HON. J. H. DRUMMOND, LL. D., Portland, Me.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let 100%=original daily capital.

$100\% \times 1.05 = 105\%$=capital at beginning of second day.

$105\% \times .95 = 99.75\%$=capital at beginning of third day, and so on during the week.

$\therefore$ Saturday night he would have $100\% \times (1.05)^3 \times (95)^3 = (99.75)^3 = 99.25187\%$.

$\therefore$ $100\% - 99.25187\% = \frac{3}{4}\%$ nearly, his loss.

ALGEBRA.

130. Proposed by J. MARCUS BOORMAN, Woodmere, N. Y.

Solve $x^5 - y^5 = 2101$.....(1), $x - y = 1$.....(2). Find general formula for (1), ...(2), when $x^n - y^n = a$, $x - y = b$; for $n_0 = 3$; $n_1 = 5$; $n_2 = 7$, etc.

I. Solution by G.B.M.ZERR, A.M., Ph.D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

$(y+1)^5-y^5=2201.$

$(y^2+y)^2+y^2+y=420.$

$\therefore\ y^2+y=20 \text{ or } -21.$

$y=4 \text{ or } -5 \text{ or } -\frac{1}{2}(1\mp\sqrt{-83}).$

$x=5 \text{ or } -4 \text{ or } \frac{1}{2}(1\pm\sqrt{-83}).$

II. Solution by the PROPOSER.

$5=n,\ \triangle=1.$ $\therefore$ Table (I), below, $1^5+5xy.1^3+5x^2y^2\ (1)=2101$; so therefore $5[x^2y^2.1^1+xy.1^3]=2101-1^5.$

$\therefore\ \frac{1}{5}[2101-1^5]=420$; hence $x^2y^2+xy+\frac{1}{4}=420+\frac{1}{4}$.....(3), as $\triangle=1$, here.

$\therefore\ xy=-0.5\pm 20.5$, and $4xy=80 \text{ or } -84.$

(2).....$(x-y)^2=$ 1 1

$\therefore\ (x+y)^2=81 \text{ or } -83.$

$\frac{1}{2}$(the root of (4)+(2)= $x=\frac{1}{2}[1\pm 9]$ or $0.5[1\pm i\sqrt{83}$.....(5).

$\frac{1}{2}$(the root of (4)−(2)= $y=\frac{1}{2}[-1\pm 9]$ or $0.5[-1\pm i\sqrt{83}$, the eight roots.

That is, $x=5$ to $y=4$; $x=-4$ to $y=-5$; etc., as at (5).

Power Difference Theorems. Given $\triangle=x-y=b$, and any one of $w=xy$, or x^n-y^n, or $x^m+y^m=a$, to find all by *quadratic* to $n=5$; *cubic* to $n=9$; *quintic* to $n=11$, etc. n=odd, viz., 3...5...7...9...11, etc.; m=even integers.

(I). $x^n-y^n=\square^n+nw\square^{n-2}+[\frac{1}{2}(n-1)(n-2)-(n-2)^0]w^2\square^{n-4}+$

$n[\frac{1}{6}(n-1)(n-2)-(n-3)]w^3\square^{n-6}+n[\frac{1}{24}(n-6)^3-(n-6)]w^4\square^{n-8}+$

$n[\frac{1}{120}(n-6)\{(n-8)^3-(n-8)\}]w^5\square^{n-10}+.....$

$+\frac{1}{24}(n^3-n)w^{\frac{1}{2}(n-3)}\square^3+nw^{\frac{1}{2}(n-1)}\square.$

(II). $x^m+y^m=\square^m+mw\square^{m-2}+[\frac{1}{2}(m-1)(m-2)-(m-2)^0]w^2\square^{m-4}+.....$

(like n).....$+\frac{1}{4}(m^2)w^{\frac{1}{2}(m-2)}\square^2+2w^{\frac{1}{2}m}.$

Also solved by *H. C. WHITAKER*.

131. Proposed by HARRY S. VANDIVER, Bala, Pa.

It is well known that, when we define the symbol $\sqrt[n]{\ }a$ after the manner of elementary text-books on algebra, certain *irrational equations* may be written down which have no real or imaginary roots. Required then, the condition, if any, between a, b, c, and d such that the equation, $\sqrt{ax+b}+\sqrt{(cx^2+d)}=0$, shall have no root, real or imaginary.

I. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$(a^2-c)x^2+2abx=d-b^2,$ or $(c-a^2)x^2-2abx=b^2-d.$

$$\therefore x=\frac{1}{c-a^2}[ab\pm\sqrt{}(a^2d+b^2c-cd)].$$

If $c>a^2$ there is no root satisfying the equation.

If $c=a^2$, $x=\dfrac{d-b^2}{2ab}$.

Then if $d>b^2$ no root will satisfy the equation.

Whenever x is positive no root can be found.

II. Solution by the PROPOSER.

In connection with this problem it will be well to review the subject of irrationals somewhat.

The symbol $\sqrt[n]{a}$ where a is a positive rational number will be defined as the positive rational number satisfying the relation $x^n=a$. Making use of the theory of binomial equations, then the other roots of $x^n=a$, can be represented by $\omega_n\sqrt[n]{a}$, $\omega_n^2\sqrt[n]{a}$.....$\omega_n^{n-1}\sqrt[n]{a}$, ω_n being a primitive root of $x^n=1$.

From this $\sqrt[n]{(a+bi)}$ can be defined, a and b being rational and $i^2=-1$ for $a+bi=\rho\text{cis}\theta$ ($\rho\text{cis}\theta=\rho(\cos\theta+i\sin\theta)$ of the Argand diagram, and $\sqrt[n]{(a+bi)}=\sqrt[n]{(\rho\text{cis}\theta)}\pm\text{cis}\theta/n\sqrt[n]{\rho}$, which is intelligible since

$$\sqrt[n]{\rho}=\sqrt[n]{[\sqrt{}(a^2+b^2)]}=\sqrt[2n]{(a^2+b^2)},$$

which symbol has been defined.

The above definitions treat the irrational symbol as one-valued and agree with those given in most of the recent text-books on algebra. If we define the radical symbol as multi-valued, the irrational equation loses all its individuality and might just as well be termed a rational equation.

It is a well known fact that there are radical equations which possess no roots real or imaginary, but a thorough classification of such equations has never been given, so far as I am aware. In the course of the following solution this matter will be cleared up.

Consider the simplest possible irrational equation:

$$\sqrt{}x=1.....(1).$$

This possesses one root $x=1$, but the related equation, $\sqrt{}x=-1.....(2)$, does not possess a root, which may be rigorously demonstrated as follows.

The equation $(\sqrt{}x+1)(\sqrt{}x-1)=0.....(3)$, must contain all the roots of (1) and (2) for a value of x which reduces either factor to zero, reduces the product to zero. Expanding (3) there is obtained

$$x-1=0, \text{ or } x=1,$$

which satisfies (1) but not (2). Hence $\sqrt{}x+1=0$ possesses no root, and is the simplest irrational equation of this type.

Proceeding to the case of the equations

$$ax+b+\sqrt{(cx^2+d)}=0.....(4),$$
$$ax+b-\sqrt{(cx^2+d)}=0.,....(5).$$

Put $ax+b=y$, then (4) becomes

$$-\sqrt{c\left(\frac{y-b}{a}\right)^2+d}=+y.....(6),$$

and (5) becomes

$$\sqrt{c\left(\frac{y-b}{a}\right)^2+d}=+y.....(7).$$

If (6) is to be satisfied by a value of y, this value cannot be positive since $\sqrt{[f(y)]}+1=0$ is an impossible relation as was proved in the case of (2). The product of (6) and (7) becomes, after simplifying,

$$(c-a^2)y^2+2bcy+a^2d+b^2c=0.....(8).$$

Now it is evident that if (8) gives a positive value for y then (6) is not satisfied but (7) is; if (8) gives a negative root the reverse is the case.

In this way the following criteria are deduced, assuming that a^2d+b^2c-cd is positive. Putting

$$A=\frac{2bc}{c-a^2}, \text{ and } B=\frac{a^2d+b^2c}{c-a^2}.$$

If $A>0$ and $B>0$, then (4) has no roots and (5) has two roots.

If $A>0$ and $B<0$, or $A<0$ and $B<0$, then (4) has one root and (5) has one root.

If $A<0$ and $B>0$, then (4) has two roots and (5) has no roots.

This solves the problem as originally stated. In an analogous way one could deduce criteria for the solubility of irrational equations of any type whatever.

This problem was also discussed by J. M. Boorman, who arrived at the conclusion that there is no condition for the equation to have no roots. He maintains that Newton's (his) theorem "Every algebraic equation has as many roots as it has dimensions," holds good. But this is not true. D'Alembert's theorem does not apply in this case at all since the equation is not a rational algebraic equation. It may not be out of place to state here D'Alembert's theorem in full: "Every rational, integral, algebreic equation whose coefficients are imaginary quantities, or, in special cases, real quantities, has at least one root." When this theorem is proved it follows as a corollary that there are as many roots as there are units in the number expressing the degree of the equation. Ed.

132. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve $2^x+3^y=4$; $5^x+6^y=7$.

Solution by H. C. WHITAKER, C. E., Ph. D., Manual Training School, Philadelphia, Pa.; MARCUS BAKER, U. S. Geological Survey, Washington, D. C.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Plotting the curves represented by the two equations, they are seen to intersect at (0, 1) and also in the neighborhood of (.6, .8).

$$\frac{\log(7-6^y)}{\log 5}-\frac{\log(4-3^y)}{\log 2}=0.$$

By Double Position the roots near .6, .8 are found to be $x=.565585$, $y=.841307$; these with the values $x=0$, $y=1$ seem to be the only real roots.

Also solved by *JOHN G. KELLER*, State Normal School, Albany, N. Y.
The Proposer sent results only, to-wit: $x=.56556312$, $y=.8413092$.

GEOMETRY.

123, Proposed by P. C. CULLEN, Indianola, Iowa.

If the bisectors of the base angles of a triangle are equal the triangle is isosceles.

I. Direct Demonstration by G. I. HOPKINS, A. M., Professor of Physics and Astronomy, High School, Manchester, N. H.

CONSTRUCTION. Make $\angle BDK=\angle FAB$, and draw through B the line NK, making $\angle DBK=\angle AFB$. Draw the perpendiculars KP and AN, and join A and K.

DEMONSTRATION. Since angles FAB and AFB are two angles of a triangle, DK and NK will meet.

$\therefore$ triangles DBK and AFB are equal.

$\therefore$ $DK=AB$ and $BK=BF$.

$\angle AHB=\angle ADH+\angle DAH=\angle ADH+\angle BDK$.

$\therefore$ $\angle AHB=\angle ADK$. Also $\angle AHB=\angle HBF+\angle HFB=\angle HBA+\angle HBK$.

$\therefore$ $\angle AHB=\angle ABK$. $\therefore$ $\angle ADK=\angle ABK$.

$\therefore$ $\angle KDP=\angle ABN$. $\therefore$ $\triangle DPK=\triangle ABN$.

$\therefore$ $AN=KP$, and $NB=DP$.

$\therefore$ $\triangle APK=\triangle AKN$.

$\therefore$ $AP=NK$, and $\therefore$ $AD=BK$. $\therefore$ $AB=FB$. $\therefore$ $\triangle ADK=\triangle AFB$.

$\therefore$ $\angle ABD=\angle FAB$. $\therefore$ $\angle DAB=\angle ABF$.

$\therefore$ $AC=BC$, and the triangle ABC is isosceles. Q. E. D.

II. Demonstration by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, University of Chicago, Chicago, Ill.

Let d_a be the length of the bisector of the angle A opposite to side a. Expressing the area of the given triangle and the areas of the triangles into which it is divided by the bisector of angle A, we get

$$\tfrac{1}{2}bc\sin A=\tfrac{1}{2}bd_a\sin\tfrac{1}{2}A+\tfrac{1}{2}cd_a\sin\tfrac{1}{2}A.$$

$$\therefore\ d_a=\frac{2bc}{b+c}\cos\tfrac{1}{2}A=\frac{2\sqrt{[s(s-a)bc]}}{b+c},\ \text{where } 2s=a+b+c.$$

Hence, if $d_a=d_b$, $\dfrac{2\sqrt{[s(s-a)bc]}}{b+c}=\dfrac{2\sqrt{[s(s-b)ac]}}{a+c}$.

$$\therefore\ 2(a+c)^2(s-a)b=2(b+c)^2(s-b)a.$$

$$\therefore\ (a+c)^2(b+c-a)b-(b+c)^2(a+c-b)a=0.$$

$$\therefore\ (a-b)[c^3+c^2(a+b)+3abc+ab(a+b)]=0.$$

Since the second factor is positive, it does not vanish. Hence $a=b$.

We have finally received an elementary direct proof of this theorem. The proof by Professor Hopkins is, I believe, without a flaw, and is the proof so long sought for by a number of mathematicians, among whom was Isaac Todhunter. This demonstration of Professor Hopkins' was examined by one of the ablest mathematicians in this country, and was pronounced by him to be correct. The trigonometric proof by Dr. Dickson is also flawless. We are glad to publish both of these proofs since the demonstration which we gave in Vol. VII, page 223, has been assailed.

Last January a year ago, we received an indirect proof from Prof. A. Anderson, of the University of North Carolina. Professor Anderson's demonstration is free from error and is substantially the proof given in Dr. Halsted's Synthetic Geometry, page 44, though Professor Anderson's figure is drawn quite differently from the one in Dr. Halsted's Geometry.

We shall be pleased to receive, from our readers, opinions on the above demonstrations.

153. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If P, P', Q, Q' be the extremities of two chords of a conic section, and both chords pass through the point A, show that the sum of the squares of the reciprocals of AP, AP', AQ, AQ' is constant.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

$$\frac{1}{AP}=\frac{1+e\cos\theta}{l},\quad \frac{1}{AP'}=\frac{1+e\cos(\pi+\theta)}{l}=\frac{1-e\cos\theta}{l}.$$

$$\frac{1}{AQ}=\frac{1+e\cos(\frac{1}{2}\pi+\theta)}{l}=\frac{1-e\sin\theta}{l}.$$

$$\frac{1}{AQ'}=\frac{1+e\cos(3\pi/2+\theta)}{l}=\frac{1+e\sin\theta}{l}.$$

$$\therefore\frac{1}{(AP)^2}+\frac{1}{(AP')^2}+\frac{1}{(AQ)^2}+\frac{1}{(AQ')^2}$$

$$=\frac{(1+e\cos\theta)^2+(1-e\cos\theta)^2+(1-e\sin\theta)^2+(1+e\sin\theta)^2}{l^2}=\frac{2(2+e^2)}{l^2}=\text{a constant}.$$

158. Proposed by JOHN MACNIE, A. M., Professor of Latin, University of North Dakota.

Show by a simple diagram that:

(a) If the angle-sum of an equilateral triangle is constant, that constant is a straight angle.

(*b*) If the angle-sum is less than a straight angle, the sum increases as the triangle grows less.

(*c*) If the angle-sum is greater than a straight angle, the sum decreases as the triangle grows less.

Solution by the PROPOSER.

Join the mid-points a, b, c, of the sides of an equilateral triangle ABC. The triangles with vertices A, B, C, being equal, we have $ab=bc=ac$, and $\angle Bac = \angle Cab = \angle Abc$, etc.

I. If $\angle A = \angle bac$, then $\angle A + \angle Abc + \angle Acb = \angle bac + Bac + \angle Cab = 180°$, and it is easily shown that $\triangle abc = \triangle Abc$, etc.

Lemma to II and III.

If MN be a line moving towards coincidence with BC, so as always to cut off equal parts on AB, AC; then, according as the angle-sum of ABC is $<$ or $>$ 180°, so will that of $\triangle Abc$ be $<$ or $>$ 180°.

For otherwise, as the angle-sum of Abc would vary from $>$ or $<$ 180° to $<$ or $>$ 180°, there would be some position of MN in which the angle-sum of Abc would be 180°, with consequences incompatible with the hypothesis.

II. If $\angle A < 60°$, then $\angle A + \angle Abc + \angle Acb < 180° < \angle abc + \angle Abc + \angle Cba$.
$\therefore \angle abc > \angle A$.

III. If $\angle A > 60°$, then $\angle A + \angle Abc + \angle Acb > 180° > \angle abc + \angle Abc + \angle Cba$.
$\therefore \angle abc < \angle A$.

162. Proposed by J. D. PALMER, Providence, Ky.

Given the distances from the vertices of a triangle, ABC, to the center of the in-circle, to construct the triangle.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

AO, BO, $CO = a$, b, c, respectively, where O is the center of the in-circle; BC, AC, $AB = x$, y, z, respectively. Let O_1 be the center of the ex-circle opposite A. Then

$$AO^2 = \frac{(p-x)^2}{\cos^2 \frac{1}{2}A}, \text{ where } p = \tfrac{1}{2}(x+y+z).$$

$$\therefore AO^2 = \left(\frac{p-x}{p}\right) yz = yz - \frac{xyz}{p} = yz - 4Rr = a^2.$$

$$\text{Similarly, } BO^2 = \left(\frac{p-y}{p}\right) xz = xz - 4Rr = b^2.$$

$$CO^2 = \left(\frac{p-z}{p}\right) xy = xy - 4Rr = c^2.$$

$$\therefore AO.BO.CO=abc=\frac{\triangle xyz}{p^2}=4Rr^2.$$

$$\therefore 4Rr=abc/r, \text{ and } (a^2r+abc)/r=yz,\ (b^2r+abc)/r=xz,\ (c^2r+abc)/r=xy.$$

$$\therefore x(a^2r+abc)/r=y(b^2r+abc)/r=z(c^2r+abc)/r$$

$$=\sqrt{[(a^2r+abc)(b^2r+abc)(c^2r+abc)/r^3]}.$$

$$\text{Also } a^2x+b^2y+c^2z=xyz\left(\frac{p-x}{p}+\frac{p-y}{p}+\frac{p-z}{p}\right)=xyz=1.$$

$$\therefore \frac{a^2}{yz}+\frac{b^2}{xz}+\frac{c^2}{xy}=1, \text{ or } \frac{ar}{ar+bc}+\frac{br}{br+ac}+\frac{cr}{cr+ab}.$$

$$\therefore 2abcr^2+(a^2b^2+a^2c^2+b^2c^2)r^2=a^2b^2c^2.$$

$$\therefore r^3+Ar^2=B. \quad \text{Let } r=S-\tfrac{1}{3}A.$$

$$\therefore S^3-\tfrac{1}{3}A^2S=B-2A^3/27.$$

$$\therefore S=\left(\tfrac{1}{2}B-\tfrac{1}{27}A^3+\sqrt{\frac{B^2}{4}-\frac{A^3B}{27}}\right)^{\frac{1}{3}}+\left(\tfrac{1}{2}B-\tfrac{1}{27}A^3-\sqrt{\frac{B^2}{4}-\frac{A^3B}{27}}\right)^{\frac{1}{3}}.$$

This determines r and therefore x, y, z, the sides of the triangle.

Otherwise, draw AO and produce AO to O_1 so that $OO_1=bc/r$. Upon OO_1 as diameter describe a circle. With O as center and b as a radius describe an arc cutting the circle in B. Similarly, with O as center and c as radius, draw an arc cutting the circle in C. Join BC, AC, AB, then ABC is the triangle required. For O_1 is the ex-center opposite A by construction as follows:

$$AO.AO_1=yz=(AO^2r+AO.BO.CO)/r.$$

$$\therefore AO_1=(AOr+BO.CO)/r.$$

The solution published in the last issue contained an error in the fourth line, and this vitiated the whole solution. Ed.

163. Proposed by J. C. NAGLE, Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Tex.

Given the equal sides of an isosceles triangle and the radius of the inscribed circle to solve the triangle. As a numerical example let the known sides be 27 and the radius of the inscribed circle 3.5. The problem occured in connection with some mill-work and the exterior angles of the triangle were required in order to make patterns for iron braces.

I. Solution by A. H. HOLMES, Professor of Mathematics, Bowdoin College, Brunswick, Me.

Let ABC be the triangle, O the center of the inscribed circle, and OD the perpendicular from O to the side AB, its radius; let $\angle BOD=\theta$ and $\angle AOD=\phi$. Then we have $\tan\theta+\tan\phi=\frac{5}{7}\frac{4}{}$. But $\phi+2\theta=180°$.

$$\therefore \tan\phi=-\tan 2\theta=-\frac{2\tan\theta}{1-\tan^2\theta}.$$

$$\therefore \tan^3\theta-\tfrac{5}{7}\tfrac{4}{}\tan^2\theta+\tan\theta+\tfrac{5}{7}\tfrac{4}{}=0.$$

From this equation we find $\tan\theta=1.16425$.

$\therefore \angle ABC=\angle ACB=81° 19' 14''$, and $\angle BAC=17° 21' 32''$.

We now readily find $BC=8.149+$.

II. Solution by H. C. WHITAKER, C. E., Ph. D., Manual Training School, Philadelphia, Pa.; G.B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

$$r=(s-a)\tan\tfrac{1}{2}A=\tfrac{1}{2}(b+c-a)\sqrt{\frac{(a+c-b)(a+b-c)}{(a+b+c)(b+c-a)}}.$$

But $b=c$. $\therefore r=a\sqrt{\frac{2b-a}{2b+a}}$.

$$\therefore a^3-2a^2b+4ar^2+8br^2=0.$$

This cubic can be solved, and gives the values of the unknown sides.

Substituting values, we get $a^3-54a^2+49a+2646=0$.

$\therefore a=8.1498$, 52.0838, or -6.2336. The first and second values are both admissible. $\cos B=a/2b=.15092$ or $.96451$.

$\therefore B=81° 19'$, or $15° 18'$.

Also solved by J. SCHEFFER.

We received a solution of problem 156 from Prof. Henry Heaton too late for credit in the number in which the solution of the problem was published. Professor Heaton gave construction also when the point does not lie between the two parallels. As the construction is similar to the case when the point lies between the two parallels we will not publish the construction.

CALCULUS.

REMARKS ON THE SOLUTION OF PROBLEM 123.

By Wm. E. Heal, Marion, Ind.

A remarkable error occurs in the published solution of Problem 123, in writing $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots=\frac{\pi^3}{35}$.

The summation referred to by Professor Echols and the Editor is incorrectly quoted. The true formula is

$$\tau_{2m+1}=\frac{E_m\pi^{2m+1}}{2^{2(m+1)}(2m)!}=1-\frac{1}{3^{2m+1}}+\frac{1}{5^{2m+1}}-\ldots\ldots$$

See Chrystal's *Algebra*, Part II, page 342. For $m=1$ this becomes, since $E_1=1$,

$$\tau_3 = \frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\ldots$$

I believe the summation of

$$\tau_2 = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots\ldots$$

is impossible in finite terms. In other words the integral

$$\int_0^{\frac{1}{4}\pi} \log(\tan z)\, dz$$

is a fundamental form. This integral is given in the great "Tables d'Intégrales Définies" of Bierens de Haan, Table 286, No. 11, in the form

$$\int_0^{\frac{1}{4}\pi} lTgx\, dx = -\sum_0^{\infty} \frac{(-n)^n}{(2n+1)^2}.$$

Although the summation in the second member constantly occurs in the above named work, no finite expression is to be found. This is conclusive evidence that no finite expression was known to the author since his results are given in finite terms when possible. For example, in the same table, No. 16, is given

$$\int_0^{\frac{1}{4}\pi} (lTgx)^2\, dx = \frac{\pi^3}{16}.$$

Chrystal's *Algebra* gives finite summations for

$$\sigma_{2m} = 1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \ldots\ldots$$

$$\sigma'_{2m} = 1 + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \frac{1}{7^{2m}} + \ldots\ldots$$

$$\sigma''_{2m} = 1 - \frac{1}{2^{2m}} + \frac{1}{3^{2m}} - \frac{1}{4^{2m}} + \ldots\ldots$$

$$\tau_{2m+1} = 1 - \frac{1}{3^{2m+1}} + \frac{1}{5^{2m+1}} - \frac{1}{7^{2m+1}} + \ldots\ldots$$

But neither in that work nor in Jordan's "Traité d'Analyse," nor in the very elaborate "Traité d'Analyse" of Laurent have I been able to find the summation of τ_{2m}. If a finite summation for τ_{2m} exists it seems incredible that it should have escaped the notice of these writers.

REMARK. On again looking up our reference we find that we quoted the reference correctly, though as Mr. Heal observes, it seems that our reference is wrong. The value of the series

$$1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\text{etc.}=\frac{\pi^3}{32}$$

is also given in Carr's *Synopsis of Pure Mathematics*, page 432. Since it now appears that the *value* of the integral is not known in finite terms, we will renew our offer to give the person finding such value a year's subscription to the MONTHLY. ED.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Show that the *complete primitive* of the differential equation

$$\left[\tan^{-1}(x)-\frac{x}{1+x^2}\right]\frac{dy^2}{dx^2}=1\left[\frac{x}{(1+x^2)^2}\right]\left[x\frac{dy}{dx}-y\right]$$

is $y=C\tan^{-1}(x)+cx$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

w being one solution of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0.....(1),$$

and P, Q functions of x, $y=vw.....(2)$ is given, by the usual theory, from

$$\frac{d^2v}{dx^2}+\left(\frac{2}{w}\frac{dw}{dx}+P\right)\frac{dv}{dx}=0.....(3),\text{ or } v=c_2+c_1\int\frac{dx}{w^2}e^{-\int P dx}.....(4).$$

In the given equation, $w=x.....(5)$, and

$$P=-\frac{2x^2}{(1+x^2)^2}\div\left[\tan^{-1}x-\frac{x}{1+x^2}\right].....(6).$$

(4) now easily becomes

$$v=c_2+c_1\int\frac{dx}{x^2}\left(\tan^{-1}x-\frac{x}{1+x^2}\right)=c_2+c_1\left(\int\frac{\tan^{-1}x}{x^2}-\int\frac{dx}{x(1+x^2)}\right)$$

$$=c_2-c_1\frac{\tan^{-1}x}{x^2}.....(7),$$

and (2) is $y=c_2x-c_1\tan^{-1}x.....(8)$.

II. Solution by W. W. BEMAN, A. M., Professor of Mathematics, State University, Ann Arbor, Mich.

$$\left(\tan^{-1}x-\frac{x}{1+x^2}\right)\frac{d^2y}{dx^2}=\frac{2x}{(1+x^2)^2}\left(x\frac{dy}{dx}-y\right).$$

Writing the equation in the form

$$\frac{d}{dx}\left(\frac{\tan^2 x}{x}\right)\cdot\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\cdot\frac{d}{dx}\left(\frac{y}{x}\right)$$

it is obvious that $y=\tan^{-1}x$ and $y=x$ are independent particular integrals. Hence the complete primitive is $y=c_1\tan^{-1}x+c_2x$.

Solutions of this problem were also received from *L. C. WALKER*, and *G. B. M. ZERR*.

126. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find the volume contained between the conical surface whose equation is $z=a-\sqrt{(x^2+y^2)}$, and the planes whose equations are $x=z$ and $x=0$ by the formula $\int\int\int dxdydz$. [Todhunter's *Integral Calculus.*]

Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

The cone clearly extends from vertex $(0, 0, a)$ towards $z=0$. Hence in

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}dxdydz$$

we have $z_1=x$; $z_2=a-\sqrt{(x^2-y^2)}$; $x=a-\sqrt{(x^2+y^2)}$.

$\therefore\ y^2=a^2-2ax$, $x_1=0$, and for x_2 we have $y_1=y_2$. $\therefore\ x=\frac{1}{2}a$.

$$\therefore\ \text{Volume}=\int_0^{\frac{1}{2}a}\int_{y_1}^{y_2}[a-x-\sqrt{(x^2+y^2)}]dxdy$$

$$=\int_0^{\frac{1}{2}a}\left[(a-x)(y_2-y_1)-\frac{y}{2}\sqrt{(x^2+y^2)}-\frac{x^2}{2}\log y+\sqrt{(x^2+y^2)}\right]_{y_1}^{y_2}dx$$

$$=\int_0^{\frac{1}{2}a}\left[2(a-x)\sqrt{(a^2-2xa)}-\sqrt{(a^2-2xa)}(a-x)-\frac{x^2}{2}\log\frac{a-x+\sqrt{(a^2-2ax)}}{a-x-\sqrt{(a^2-2ax)}}\right]dx.$$

Put $2x=a\sin^2\varphi$.

$$\therefore\ V=\int_0^{\frac{1}{2}\pi}a\sin\phi\cos\phi d\phi\left[(1+\cos\phi)\frac{a^2}{2}\cos\phi-\frac{a^2}{8}\sin^4\phi\log\left(\frac{1+\cos\phi}{1-\cos\phi}\right)^2\right]$$

$$=-\frac{a^3}{2}\int_0^{\frac{1}{2}\pi}\left[\sin\phi\cos^2\phi+\sin\phi\cos^4\phi-\sin^5\phi\cos\phi\log\frac{1+\cos\phi}{\sin\phi}\right]d\phi$$

$$=\frac{a^3}{2}(\tfrac{1}{3}+\tfrac{1}{5})-\left[\frac{a^3\sin^6\phi}{12}\log\frac{1+\cos\phi}{\sin\phi}\right]_0^{\frac{1}{2}\pi}-\frac{a^3}{12}\int_0^{\frac{1}{2}\pi}\sin^6\phi\left(\frac{\cos\phi}{\sin\phi}+\frac{\sin\phi}{1+\cos\phi}\right)dx$$

$$={}_{15}^{4}a^3-{}_{12}^{1}a^3\int_0^{\frac{1}{2}\pi}[\cos\phi\sin^5\phi+\sin^5\phi(1-\cos\phi)]d\phi={}_{15}^{4}a^3-\frac{a^3}{12}\frac{4.2}{5.3}=\tfrac{2}{9}a^3.$$

Similar solutions were received from *GEORGE LILLEY*, *LON C. WALKER*, *G. B. M. ZERR*, and *J. SCHEFFER*.

MECHANICS.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Develop the Fourier Series to represent the temperature of a circular wire of uniform cross-section, in which the temperatures of the four quadrants are in order $t,2t,3t,4t$.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

From Fourier's *Analytical Theory of Heat*, page 217, we get for the complete solution of the linear and varied movement of heat in a ring after a time T the following:

$$v=e^{-hT}\left[\frac{1}{2\pi}\int_0^{2\pi}f(x)dx+\frac{1}{\pi}\sum_{n=1}^{n=\infty}e^{-n^2kT}\sin nx\int_0^{\frac{1}{2}\pi}\sin nxf(x)dx\right.$$

$$\left.+\frac{1}{\pi}\sum_{n=1}^{n=\infty}e^{-n^2kT}\cos nx\int_0^{\frac{1}{2}\pi}\cos nxf(x)dx\right],$$

$$\text{where } f(x)=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx+\frac{1}{\pi}\sum_{n=1}^{n=\infty}\sin nx\int_0^{2\pi}\sin nxf(x)dx$$

$$+\frac{1}{\pi}\sum_{n=1}^{n=\infty}\cos nx\int_0^{2\pi}\cos nxf(x)dx.$$

$$\int_0^{2\pi}f(x)dx=t\int_0^{\frac{1}{2}\pi}dx+2t\int_{\frac{1}{2}\pi}^{\pi}dx+3t\int_{\pi}^{\frac{3}{2}\pi}dx+4t\int_{\frac{3}{2}\pi}^{2\pi}dx=5\pi t.$$

$$\int_0^{2\pi}\sin nxf(x)dx=t\int_0^{\frac{1}{2}\pi}\sin nxdx+2t\int_{\frac{1}{2}\pi}^{\pi}\sin nxdx+3t\int_{\pi}^{\frac{3}{2}\pi}\sin nxdx+4t\int_{\frac{3}{2}\pi}^{2\pi}\sin nxdx$$

$$=\frac{t}{n}(\cos\tfrac{1}{2}\pi n+\cos\pi n+\cos\tfrac{3}{2}\pi n)-\frac{3t}{n}=-\frac{4t}{n},\text{ except when } n=4m.$$

$$\int_0^{2\pi}\cos nxfxdx=t\int_0^{\frac{1}{2}\pi}\cos nxdx+2t\int_{\frac{1}{2}\pi}^{\pi}\cos nxdx+3t\int_0^{\frac{3}{2}\pi}\cos nxdx+4t\int_{\frac{3}{2}\pi}^{2\pi}\cos nxdx$$

$$= -\frac{t}{n}(\sin\tfrac{1}{2}\pi n + \sin\tfrac{3}{2}\pi n) = 0.$$

$$\therefore f(x) = \tfrac{5}{2}t - (t/\pi)(4\sin x + 2\sin 2x + \tfrac{4}{3}\sin 3x + \tfrac{4}{5}\sin 5x + \tfrac{4}{6}\sin 6x + \tfrac{4}{7}\sin 7x + \tfrac{4}{9}\sin 9x + \ldots)$$

$$\therefore V = e^{-hT}[\tfrac{5}{2}t - (t/\pi)(4\sin x e^{-kT} + 2\sin 2x e^{-2^2kT} + \tfrac{4}{3}\sin 3x e^{-3^2kT} + \tfrac{4}{5}\sin 5x e^{-5^2kT} + \ldots)]$$

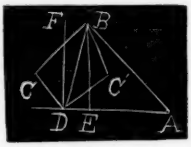
AVERAGE AND PROBABILITY.

108. Proposed by A. H. HOLMES, Brunswick, Me.

Required the average area of the quadrilateral whose sides are a, b, c, and d.

I. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ABCD$ be the quadrilateral, $AB=a$, $AD=b$, $BC=c$, $CD=d$, $a>b>c>d$, $BE=v$, $DE=u$,.



When the quadrilateral is convex, the average area is, (the average area of ABD) + (the average area of BCD).

When the quadrilateral is concave, the average area is, (the average area of ABD) − (average area of $BC'D$).

Since BCD is equal to $BC'D$, the average area required is the average area of $ABD = \triangle$. Area $ABD = \frac{1}{2}ab\sin A$.

Now $DB = \sqrt{(u^2+v^2)}$.

$\therefore \cos A = (a^2+b^2-u^2-v^2)/2ab$.

$\therefore$ Area $ABD = \frac{1}{4}\sqrt{[4a^2b^2-(a^2+b^2-u^2-v^2)^2]}$.

But $v^2 = a^2-(b-u)^2$.

$\therefore$ Area $ABD = \frac{1}{2}b\sqrt{[a^2-(b-u)^2]} = C$.

The limits of u are 0 and $[(c+d)^2+b^2-a^2]/2b = u'$.

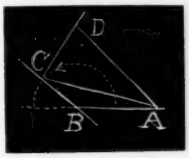
$$\therefore \triangle = \frac{\int_0^{u'} C\,du}{\int_0^{u'} du} = \frac{b}{2u'}\int_0^{u'}\sqrt{[a^2-(b-u)^2]}\,du.$$

$$\therefore \triangle = \frac{b^2}{(c+d)^2+b^2-a^2}\left[\frac{b}{2}\sqrt{(a^2-b^2)} + \frac{a^2}{2}\sin^{-1}\frac{b}{a} - \frac{a^2+b^2-(c+d)^2}{8b^2}\right.$$

$$\left.\sqrt{\{4a^2b^2-[a^2+b^2-(c+d)^2]^2\}} - \frac{a^2}{2}\sin^{-1}\left(\frac{a^2+b^2-(c+d)^2}{2ab}\right)\right].$$

II. Solution by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, O.

Let $ABCD$ be the quadrilateral, side $AB=a$, $BC=b$, $CD=c$, $DA=d$, $\angle ABC=\theta$, and $\angle CDA=\phi$.



Suppose the vertices of the quadrilateral to be movable, or hinged; then the side BC may make a complete revolution about B as a center so long as $(c+d)$ is not less than $(a+b)$. That is, for $(c+d)$ not less than $(a+b)$, the angle θ will vary uniformly from 0° to 360°; but as soon as the side BC has moved below the side AB produced, we have hour-glass quadrilaterals composed of two practically isolated triangles which are not to be considered in finding the average area of the quadrilateral (a, b, c, d). We are, therefore, constrained to regard θ as varying uniformly from 0° to 180°; but if $(c+d)<(a+b)$, which becomes certain when the sides of the quadrilateral (a, b, c, d) are numerically expressed, we are constrained to regard θ as varying uniformly from 0° to $\cos^{-1}\{[a^2+b^2-(c+d)^2]/2ab\}$.

From the diagram it is evident that $Q=(\triangle ABC+\triangle CDA)=\frac{1}{2}(ab\sin\theta+cd\sin\phi)$. Now $AC=\sqrt{(a^2+b^2-2ab\cos\theta)}$, and

$$\cos\phi=\frac{c^2+d^2-(a^2+b^2-2ab\cos\theta)}{2cd}.$$

[For the sake of brevity, put $m=c^2+d^2-a^2-b^2$, $n=4c^2d^2-m^2$, $p=n/4a^2b^2$, and $q=m/ab$].

$$\therefore \sin\phi=\sqrt{1-\left(\frac{m+2ab\cos\theta}{2cd}\right)^2}=\sqrt{\frac{(4c^2d^2-m^2)-4abm\cos\theta-4a^2b^2\cos^2\theta}{4c^2d^2}}$$

$$=\frac{1}{2cd}\sqrt{(n-4abm\cos\theta-4a^2b^2\cos^2\theta)}=(ab/cd)\sqrt{(p-q\cos\theta-\cos^2\theta)}$$

$$=(ab/cd)\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}.$$

Also, $Q=\frac{1}{2}ab\{\sin\theta+\sqrt{[(p+\frac{1}{4}q^2)-(\frac{1}{2}q+\cos\theta)^2]}\}$.

Representing $\cos^{-1}\{[a^2+b^2-(c+d)^2]/2ab\}$ by θ_1, the expression for the average area of the quadrilateral on the hypothesis that the interior angle at B vary uniformly from 0 to θ_1 becomes

$$Q_B=\frac{ab}{2\theta_1}\int_0^{\theta_1}\{\sin\theta+\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}\}d\theta.$$

Similar operations give Q_C, Q_D, and Q_A; and, therefore, the required average area of the quadrilateral (a, b, c, d) becomes $\mathbf{Q}=\frac{1}{4}(\mathbf{Q}_A+\mathbf{Q}_B+\mathbf{Q}_C+\mathbf{Q}_D)$. Slight modifications, in signs, etc., may be occasioned by quadrilaterals having a re-entrant angle.

MISCELLANEOUS.

104. Proposed by HARRY S. VANDIVER, Bala, Pa.

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. Chrystal's *Algebra*, Vol. II., page 535.]

Solution by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Since the sides of the right-angled triangle may have a common factor, we let af, bf, and cf represent its sides, where a, b, c are prime to each other. Then we have $a^2+b^2=c^2$.....(1).

For the area, $\frac{1}{2}abf^2$, of the triangle to be a square, $\frac{1}{2}ab$ must be a square.

In (1) either a or b must be even and the other odd, because the sum of two odd squares can not be a square. Assume a even and b odd. Then for $\frac{1}{2}ab$ to be a square, b must be a square, and a must be double a square.

Let $a=2m^2$, $b=n^2$, and substitute these values in (1); then we have

$$(2m^2)^2+(n^2)^2=c^2.....(2).$$

Now set $2m^2=2pq$, and $n^2=p^2-q^2$; or

$$n^2+q^2=p^2.....(3).$$

In (3) p is odd, and we will assume q even. Let $q=2\alpha\beta$, $p=\alpha^2+\beta^2$, and we find $m=2\alpha\beta(\alpha^2+\beta^2)$.....(4).

For $2\alpha\beta$ to be a square, we will assume $\alpha=2m_1^2$ and $\beta=n_1^2$. Substituting these values in (4), we obtain

$$m^2=4m_1^2n_1^2(4m_1^4+n_1^4).....(5).$$

In order that the right member of (5) may be a square, we must have

$$4m_1^4+n_1^4=c_1^2,$$

say; which is of the same form of (2). But c_1^2 is less than c^2. Proceeding in exactly the same way, we can reduce c^2 indefinitely. By our hypothesis c^2 can not be reduced to zero nor less. Hence, results *Fermat's Theorem.*

PROBLEMS FOR SOLUTION.

ARITHMETIC.

154. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Suppose there is a meadow of 8 acres in which the grass grows uniformly, and that 21 oxen could eat up the whole pasture in 6 weeks, or 18 oxen in 9 weeks; what number of oxen diminished by the removal of 9, at the end of 14 weeks, could eat it up in 18 weeks?

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A bought a horse, which he sold to B at a *loss* of $m=6\%$; B sold the horse to C at a *loss* of $n=5\%$; and C sold the horse to D at a *gain* of $p=12\frac{1}{2}\%$. How much did A *lose*, if C gained $\$G=\$26.79$?

ALGEBRA.

156. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

$(z+x)a-(z-x)b=2yz$.....(1); $(x+y)b-(x-y)c=2xz$.....(2); $(y+z)c-(y-z)a=2xy$.....(3). Find the values of x, y, and z, by the method of linear simultaneous equations.

NOTE. This problem was somewhat abbreviated in the last issue. The problem occurs in Fisher & Schwatt's *Elements of Algebra*, page 224, under Linear Simultaneous Equations.

157. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Solve the equations,

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}+\frac{u}{d+\lambda}=1, \qquad \frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}+\frac{u}{d+\mu}=1,$$

$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}+\frac{u}{d+\nu}=1, \qquad \frac{x}{a+\rho}+\frac{y}{b+\rho}+\frac{z}{c+\rho}+\frac{u}{d+\rho}=1.$$

158. Proposed by R. D. BOHANNON, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1,$

show, $\frac{x}{(a+\beta)^2}+\frac{y}{(b+\beta)^2}+\frac{z}{(c+\beta)^2}=\frac{(\gamma-\beta)(\alpha-\beta)}{(a+\beta)(b+\beta)(c+\beta)}.$

GEOMETRY.

182. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Show how to cut from a given cube, edge s, the maximum tetrahedron.

183. Proposed by S. F. NORRIS, Professor of Mathematics and Astronomy, Baltimore City College, Baltimore, Md.

"Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures." [Olney's *Geometry*, page 129.]

CALCULUS.

148. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Hemholtz's differential equation for the strength of an electric current C at any time t, is $C=E/R-L/R\times dC/dt$. Solve this equation, supposing $C=0$ when $t=0$; and E, R, L are to be regarded as constants.

149. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the volume contained between the plane $z=(a-x)\cot\beta$ and the surface $xz^2=(a-x)(x^2+y^2)$.

MECHANICS.

138. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

A smooth elliptical tube is held in the vertical plane with its major axis inclined to the vertical. A particle is projected from the lowest point. Find the pressure on the tube at any point and the condition that the pressure may vanish at the highest point.

139. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A homogeneous sphere, radius $r=50$ inches, makes $m=30$ revolutions around an axis every second. The mass begins to disappear from the surface into space at a rate exactly sufficient to cause the diameter to decrease uniformly at the rate of $(1/n)$th$=1/1000$th of a linear inch per second. At what rate per second is the angular velocity of the sphere changing the instant the diameter becomes $p=10$ inches? What is the diameter of the sphere when the rate of disappearance of matter is midway between minimum and maximum? When is the angular velocity a maximum? How does this maximum angular velocity compare with the original angular velocity? What is the diameter of the sphere when the paracentric force is (1) a maximum and (2) a minimum?

DIOPHANTINE ANALYSIS.

101. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

If p and q are such values of x and y as fulfill the conditions $x^2\pm y^2-1=$a square, find, in terms of p and q, the expression for an indefinite number of other values.

102. Proposed by A. H. BELL, Hillsboro, Ill.

Prove that every indeterminate equation of the second degree can be reduced to $x^2-Ay^2=Bz^2$. [*Legendre.*]

AVERAGE AND PROBABILITY.

123. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Three points are taken at random within a square. What is the probability that the triangle formed by joining them is acute?

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the average area of a spherical polygon of $n=6$ sides.

MISCELLANEOUS.

124. Proposed by J. W. YOUNG, Graduate Student, Cornell University, Ithaca, N. Y.

Prove that the general value of θ, which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\ldots\text{to } n \text{ factors}=1 \text{ is } \frac{4m\pi}{n(n+1)};$$

where m is any integer ($i=\sqrt{-1}$).

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Assume $m=nt+\varepsilon-\omega$, thus giving $v=m+e\sin v$ as the relation connecting the mean and eccentric anomalies, then express $x=a\cos v$, $y=b\sin v$, and $r=a(1-e\cos v)$ by a Fourier series in terms of m.

BOOKS AND PERIODICALS.

Graphs. By Robert J. Aley, A. M., Ph. D., Professor of Mathematics in the Indiana University. Pamphlet Form, 21 pages. Price, 10 cents. Boston and Chicago: D. C. Heath & Co.

This is No. 6 of Heath's Mathematical Monographs issued under the general editorship of Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. In this monograph, Dr. Aley has pointed out the various uses that are being made of Graphs in almost every department of knowledge. In addition to plotting a temperature curve, four problems are solved by means of graphs, of which the first is the following: "A travels 4 miles an hour, B 6 miles an hour. If A has two hours the start, when and where will B overtake him?" Then comes the solution of Linear Equations in two variables, and following these, Simultaneous Quadratics, closing with the graphic representation of complex numbers.

College Algebra. By Leonard Eugene Dickson, Ph. D., Assistant Professor of Mathematics in the University of Chicago. First edition. First thousand. 8 vo. Cloth, vi+214 pages. Price, $1.50. New York: John Wiley & Sons.

"This text," the author tells us, "is intended primarily for college and technical schools. By treating only the subjects usually given in the college course in algebra, space has been gained for more detailed exposition of the more difficult topics." The work begins with a treatment of Number in Algebra; Surds and Imaginaries. Then follows in order the subjects Exponents—Logarithms; FactorTheorem—QuadraticEquations; Simultaneous Equations—Determinants; Ratio—Proportion, Variation, the Progressions, Compound Interest and Annuities; Undetermined Coefficients, Partial Fractions; Permutation and Combination, Binomial and Multinomial Theorem; Probability, Mathematical Induction, Limits, Indeterminate Forms; Convergency and Divergency of Series; Power Series and other series, Summation of Series, the Method of Differences, Graphic Algebra,

Theory of Equations and an Appendix. Every subject treated by Dr. Dickson is treated in a thoroughly scientific manner, and with that accuracy and rigor so characteristic of all of his mathematical work. Dr. Dickson deserves the highest praise for sacrificing his time so fully devoted to original investigations in the various branches of mathematics and giving it to the preparation of text-books to be used in colleges. We hope that this edition will soon be exhausted, and that in the second edition he will add considerable material on the still more elementary parts of algebra, thus adapting the work to a larger constituency.

The Public School Arithmetic. For Grammar Grades, based on McLellan and Dewey's "Psychology of Numbers." By J. A. McLellan, A. M., LL. D., and A. F. Ames, A. B. 16 mo. Cloth. xii+369 pages. Price, 60 cents. New York: The Macmillan Co.

This book is based on Dewey & McLellan's "Psychology of Numbers," the basal idea of which is the correlation of number and measurement. They hold, *no number without measurement, no measurement without number.* This idea of number has been, I think, very successfully refuted a number of times by a number of different mathematicians and educators. Certainly the idea of number existed a long time before the idea of measurement. If one were to count the number of stone in a stone fence, the unit of measure in this case is one certainly very hard to adjust to our modern notion of a unit of measure. It would seem that a book, based upon what seems to a great number, perhaps the greater number, of our best educators to be erroneous, would have very little to commend it. But this is not true in regard to the book under consideration. The problems for solution are thoroughly practical, being taken from every source of modern life, commercial, agricultural, industrial and social. These problems are presented in such a way that the pupil can easily realize the conditions which gave rise to them. The systematic arrangement of the solutions of illustrative problems is another feature of the work which is highly commendable. These solutions are given in steps, though steps are not numbered. The subject of mensuration which, because of pressure from outside, has been omitted entirely by a number of authors, has received attention in this book commensurate with its importance. There are a number of other features that are commendable, for the appreciation of which the book should be consulted.

The American Journal of Mathematics. Edited by Frank Morley with the cooperation of Simon Newcomb and other mathematicians. Published under the auspices of the Johns Hopkins University. Price, $5.00 per year in advance. No. 1, Vol. xxiv, contains to following articles:

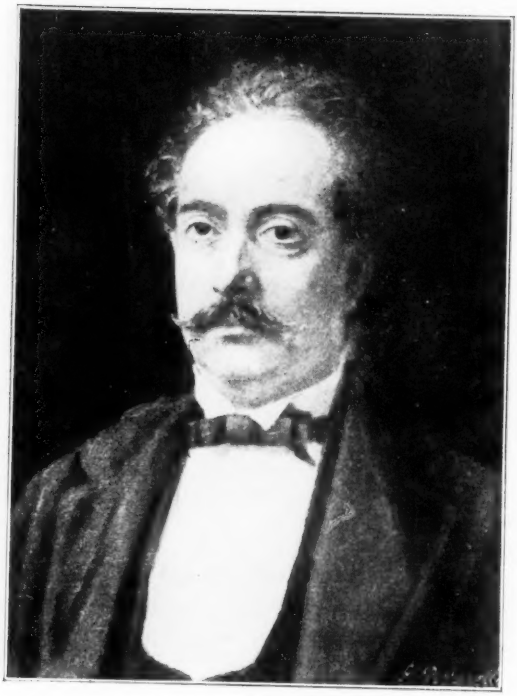
Cyclic Group of the Simple Ternary Linear Fractional Group in a Galois Field, by L. E. Dickson; Curves of Triple Curvature, by James G. Hardy; Primary Prime Functions in Several Variables and a Generalization of an Important Theorem of Dedekind, by Harris Hancock; On Certain Properties of the Plane Cubic Curve in Relation to Circular Points at Infinity, by R. A. Roberts; Estimates of Pierce's Linear Associative Algebra, by H. E. Hawkes; Groups Defined by the Orders of Two Generators and the Order of Their Product, by G. A. Miller.

Annals of Mathematics. Edited by Ormond Stone, W. E. Byerly, W. F. Osgood and others. Published quarterly under the auspices of Harvard University. Price, $2.00 per year. No. 2, Vol. 3, second series, January, 1902, contains the following articles:

Some Applications of the Method of Abridged Notation, by Professor Maxime Bocher; On Roots of Functions Connected by a Linear Recurrent Relation of the Second Order, by Dr. M. B. Porter; Space of Constant Curvature, by Professor F. S. Woods.

Cyclographic Transformation of Ordinary Space. By Dr. Arnold Emch. Reprint from the University of Colorado Studies.

The following periodicals have been received: *The Open Court*, for February; *Educational Times*, for February; *Kansas University Quarterly; Bulletin of the American Mathematical Society; Scientific American;* and *The American Monthly Review of Reviews.*



EUGENIO BELTRAMI.

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EUGENIO BELTRAMI.

By DR. GEORGE BRUCE HALSTED.

The great vindicator and interpreter of non-Euclidean geometry, Beltrami, like its two creators Bolyai and Lobachevski, was noted as a student for his j ɑbordination.

Beltrami was born at Cremona on November 16th, 1835, and attended the ɛ mentary schools, gymnasium and liceum of that city, except for the scholastic yɛ 1848-49 in which he was at that gymnasium of Venice which now bears the nɔme of Marco Polo.

Having finished the liceal studies in the summer of 1853, in the following Novɛmber he inscribed himself as student in the Mathematical Faculty of the University of Pavia, after having obtained there a place on the Castiglioni Foundation in the Collegio Ghislieri.

But in the succeeding year, accused of having promoted disorders against the Abbot Leonardi, rector of this college, he was expelled from it, together with five others of his college mates. Thus, like Lobachevski, not even being a charity beneficiary could restrain his irrepressible independence.

This expulsion worked a terrible hardship and disadpointment on the ambitious youth. As Loria says in his notice of Beltrami, on which we draw here, "This measure—perhaps not absolutely without grounds, but certainly too rigorous—had disastrous consequences for its victim."

For, though the University was still open to him, and though he had the

great good fortune to attend certain lectures of Brioschi, yet the poverty of his family constrained him to return home before having passed the examination which precedes the doctorate. In November, 1856, he went to Verona where he had obtained the humble position of secretary to the engineer Diday in the government service of Lombardy-Venice. Here he remained until the 10th of January, 1857, when *for political reasons* he was bruskly dismissed by the director-general, Busche.

Fortunately the annexation of Lombardy to Piedmont, happening soon after, allowed Diday to transfer his office to Milan, taking his secretary with him.

At Milan Beltrami undertook all over again his mathematical education. Here he was so fortunate as to have access again to his former professor, Brioschi, and also to Luigi Cremona.

The friendship of these great men was of decisive influence for his life, opening to him at a stroke the very career for which he had longed. Thus he who for lack of a degree had seen himself disbarred from the secondary schools and from the corps of military engineers, was, on the basis of his publications in the *Annali di Matematica*, named (Oct. 18th, 1862,) "Professore straordinario" in the University of Bologna.

It is to the honor of Cremona to have suggested this appointment, and of Brioschi, then secretary-general to the Minister of Public Instruction, to have adopted the suggestion. By it Beltrami was at one stroke liberated from the shackles of a humble administrative occupation and put in position to consecrate himself wholly to the genial occupation for which he was by nature so pre-eminently fitted.

The very next year, on the proposal of Enrico Betti, he was offered the professorship of geodesy in the University of Pisa. From Pisa he returned to Bologna in September, 1866, as professor of rational mechanics.

But it was ideas of a geodetic character which gave that turn to his creative genius destined to be crowned with such brilliant fame. In the exordium of a memoir dated Pisa, May 31st, 1866, Beltrami remarks that in treating of a map destined to serve for measurements of distance it would be most convenient to determine, that to the geodetics of the surface should correspond the straights of the plane, because, such a representation accomplished, the questions concerning geodetic triangles would be reduced to simple questions of plane trigonometry. He concludes that "the only surfaces capable of being represented on a plane so that to every point corresponds a point and to every geodetic a a straight are those whose curvature is everywhere constant."

Now at the very time that Beltrami was working on surfaces of constant curvature, Baltzer in Germany, Hoüel in France and Battaglini in Italy had set to work to diffuse the revolutionary ideas of Bolyai and Lobachevski, while Dedekind published the "Habilitationsvorlesung" of Riemann: "On the hypotheses which lie at the basis of geometry."

Of this Beltrami made an annotated translation of which he speaks in two letters to Genocchi: "Bologna, 9 June, 1868. * * * The manuscript of